# A new rank parity computing machine

Holly Green

University College London

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# Main results

## Theorem (Constantinou, Dokchitser, Green, Morgan)

Assume # $X$ is nite. For all smooth, projective curves over number elds $X/K$

$$\mathrm{rank}(\mathrm{Jac}_X/K) \quad \sum_v (X/K_v) \mod 2$$

where $\in \{0,1\}$ is an explicit invariant computed from curves over local elds.

## Work in progress theorem (Dokchitser, Green, Morgan)

Assume # $X$ is nite. The Birch and Swinnerton-Dyer conjecture correctly predicts the parity of $\mathrm{rank}(\mathrm{Jac}_X/K)$ for all nice hyperelliptic curves over number elds $X/K$.

## Theorem (Green, Maistret $p = 2$ and $E$ has CM)

The $p$-parity conjecture holds for elliptic curves over totally real number elds.

# BSD and the parity conjecture

## Birch and Swinnerton-Dyer conjecture

$$\mathrm{rank}(\mathrm{Jac}_X) = \mathrm{ord}_{s=1} L(\mathrm{Jac}_X; s)$$

+

## Conjectural functional equation

$$L(\mathrm{Jac}_X; s) = w(\mathrm{Jac}_X) L(\mathrm{Jac}_X; 2 \quad s)$$

## The Parity Conjecture

Let $K$ be a number eld and $X = K$ a curve. Then

$$( \quad 1)^{\mathrm{rank}(\mathrm{Jac}_X = \quad)} \qquad\qquad K$$

# Applications of local formulae

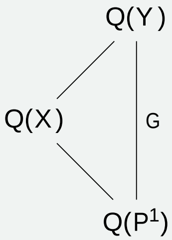
Let $E/K$ be a semistable elliptic curve. Assuming BSD, or finiteness of Ш,

$$\mathrm{rank}(E/K) \equiv \#\{v \mid \infty\} + \#\{v \nmid \infty,\ E/K_v \text{ split multiplicative}\} \mod 2.$$

- $E/\mathbb{Q}$

# Ingredient 1 for the parity computing machine: eld diagrams

Let $X/\mathbb{Q}$ be a curve and $\pi : X \to \mathbb{P}^1$.

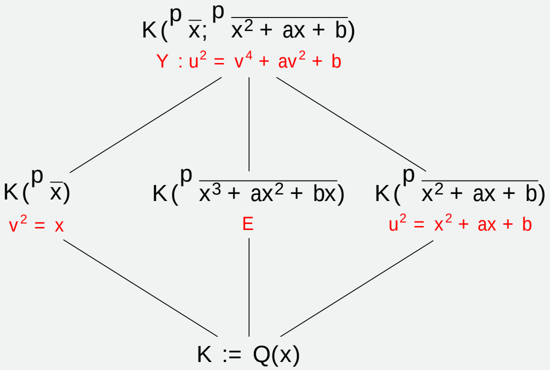



- $\mathbb{Q}(Y)^H = \mathbb{Q}(Y/H)$
- $(\mathrm{Jac}_Y(\mathbb{Q}) \otimes \mathbb{Q})^H = \mathrm{Jac}_{Y/H}(\mathbb{Q}) \otimes \mathbb{Q}$
- Tate modules
- Selmer groups
- Height pairings

## Example

Let $E : y^2 = x^3 + ax^2 + bx$ and $\pi : E \to \mathbb{P}^1$,

$$(x, y) \mapsto x.$$



# Ingredient 2 for the parity computing machine: Brauer relations

Let $G$ be a nite group.

# The parity computing machine

## Theorem (Constantinou, Dokchitser, Green, Morgan)

Let $Y/\mathbb{Q}$ be smooth, projective such that $\#\text{Ш}_{\mathrm{Jac}_Y}[\ell^\infty]$ is finite. Assume $Y \to \mathbb{P}^1$ is a Galois cover and let $\Theta = \sum_i H_i - \sum_j H_j'$ be a Brauer relation for its Galois group. Then

$$\mathrm{ord}_\ell \left( \frac{\prod_i \mathrm{Reg}_{\mathrm{Jac}_{Y/H_i}}}{\prod_j \mathrm{Reg}_{\mathrm{Jac}_{Y/H_j'}}} \right) \equiv \sum_v \Lambda_\Theta(Y/\mathbb{Q}_v) \mod 2$$

where $\Lambda_\Theta$ is an expression in $\Theta$ and local data for $Y/\mathbb{Q}_v$.

## Example

# The parity computing machine

We recover local formulae for:

- $E$ admitting a cyclic $\ell$-isogeny (Cassels), if $E(K)[\ell] \neq \{O\}$ then $D_{2\ell}$
- $\mathrm{Jac}_X$ for $X$ hyperelliptic over quadratic extensions (Kramer, Tunnell; Morgan), $C_2 \quad C_2$
- $\mathrm{Jac}_X$ for $X$ of genus 2 with a Richelot isogeny (Dokchitser, Maistret), $D_8$
- $\mathrm{Jac}_X$ for $X$ of genus 3 such that $G$ acts on $\mathrm{Jac}_X[2]$ by a $K$ 2-group (Docking). $S_4$

**Theorem (Constantinou, Dokchitser, Green, Morgan)**

Thank you for your attention!